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Linear bilevel multi-follower programming with independent followers

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Abstract This paper considers a particular case of linear bilevel programming problems with one leader and multiple followers. In this model, the followers are independent, meaning that the objective function and the set of constraints of each follower only include the leader's variables and his own variables. We prove that this problem can be reformulated into a linear bilevel problem with one leader and one follower by defining an adequate second level objective function and constraint region. In the second part of the paper we show that the results on the optimality of the linear bilevel problem with multiple independent followers presented in Shi et al. [The *k*th-best approach for linear bilevel multi-follower programming, J. Global Optim. 33, 563–578 (2005)] are based on a misconstruction of the inducible region.

Keywords Bilevel programming · Linear · Multiple followers · *k*th-best

1 Introduction

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. This model has been applied to decentralized planning problems involving a decision process with a hierarchical structure. It provides an appropriate model to hierarchical decision processes with two decision makers, the leader and the follower, each controlling part of the variables, both having their own objective function and constraints. Using the common notation in bilevel programming, it can be stated as:

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$$\begin{aligned}&\min_{x,y} \quad f_1(x,y),\\&\text{subject to:} \qquad g_i(x,y) \le 0, \quad i = 1,\dots,p,\end{aligned}$$

where y solves

$$\begin{aligned}&\min_{y} \; f_2(x,y),\\&\text{subject to:} \qquad h_j(x,y) \le 0, \quad j = 1,\dots,q,\end{aligned}$$

where $x \in \mathbb{R}^{n_1}$ are the variables controlled by the first level decision maker or leader and $y \in \mathbb{R}^{n_2}$ are the variables controlled by the second level decision maker or follower.

Due to its structure, bilevel problems are non-convex and quite difficult to deal with, even when all functions involved are linear. As a matter of fact, most papers in literature assume that the functions involved are linear or convex. Vicente and Calamai [9] and Dempe [6] provide surveys on bilevel programming that cover both the linear and the non-linear cases. Calvete and Galé [3,4] consider the case in which both objective functions are quasi-concave or linear fractional. Bard [2] and Dempe [5] are good general references on this topic, which provide applications as well as major theoretical developments.

This paper considers linear bilevel problems in which the second level of the hierarchy includes multiple followers. This means that each decision maker (the leader and the followers) controls a separate set of decision variables and tries to optimize his own objective function. The special case dealt with in this paper assumes that both the objective function and the set of constraints of each follower only include the leader's variables and his own variables. This fact implies that there is no communication between the followers, that is to say, they do not share any information. Therefore, the performance of a follower cannot be affected by the alternatives taken for the other followers. We have called these kind of followers, 'independent followers'. The paper is organized as follows. After setting the problem in Sect. 2, Sect. 3 provides the main theoretical result on optimality. It proves that the linear bilevel problem considered is equivalent to a linear bilevel problem with only one follower by defining an adequate second level objective function and constraint region. Section 4 examines the results stated in a paper by Shi et al. [8], showing that the notion of inducible region was misunderstood since it is essentially obtained by moving the first level constraints into each second level follower problem. Moreover, it shows that the problem analyzed in Ref. [8] can also be reformulated as a linear bilevel problem with one leader and one follower. This result provides us another tool to show the misconstruction of the inducible region in Ref. [8]. Finally, Sect. 5 concludes the paper with final remarks and future work.

2 Setting the linear bilevel problem with multiple independent followers (LBMIF)

The linear bilevel problem which second level of the hierarchy includes $K \ge 2$ independent followers is defined as:

LBMIF:

$$\min_{x,y_1,\ldots,y_K} \quad c^t x + \sum_{i=1}^{K} d_i^t y_i, \tag{1a}$$

$$\text{subject to:} \qquad Ax + \sum_{i=1}^{K} B_i y_i \le b_0, \tag{1b}$$

where $y_i,\ i = 1,\ldots,K$, solves

$$\min_{y_i} \quad v_i^t x + w_i^t y_i, \tag{1c}$$

$$\text{subject to:} \qquad Q_i x + D_i y_i \le b_i, \tag{1d}$$

where $x \in \mathbb{R}^{n_0}$ are the variables controlled by the leader and $y_i \in \mathbb{R}^{n_i}, i = 1,\ldots,K$, are the variables controlled by the ith follower; $c, v_i \in \mathbb{R}^{n_0}$, $d_i, w_i \in \mathbb{R}^{n_i}$, $b_0 \in \mathbb{R}^{m_0}$, $b_i \in \mathbb{R}^{m_i}$, $A \in \mathbb{R}^{m_0\times n_0}$, $B_i \in \mathbb{R}^{m_0\times n_i}$, $Q_i \in \mathbb{R}^{m_i\times n_0}$, $D_i \in \mathbb{R}^{m_i\times n_i}$, $i = 1,\ldots,K$. The superscript t means transposition.

Based on the hierarchical structure of bilevel problems and usual bilevel problem notations in Bard [2], the following are the relevant sets to the LBMIF problem:

(a) Constraint set of the LBMIF problem

$$S = \left\{(x, y_1,\ldots,y_K) : Ax + \sum_{i=1}^{K} B_i y_i \le b_0,\ Q_i x + D_i y_i \le b_i, i = 1,2,\ldots,K\right\}.$$

We assume that S is non-empty and compact.

(b) Feasible set for the ith follower for each x

$$S_i(x) = \{y_i : D_i y_i \le b_i - Q_i x\}.$$

Notice that constraints $Ax + \sum_{i=1}^{K} B_i y_i \le b_0$ should not be included since they only affect the first level decision maker.

(c) Projection of S onto the leader's decision space

$$S(X) = \left\{x : \exists (y_1,\ldots,y_K),\ Ax + \sum_{i=1}^{K} B_i y_i \le b_0, Q_i x + D_i y_i \le b_i, i = 1,2,\ldots,K\right\}.$$

(d) ith follower rational reaction set for $x \in S(X)$

$$P_i(x) = \operatorname*{argmin}_{y_i}\{v_i^t x + w_i^t y_i : y_i \in S_i(x)\}.$$

We assume that, for all decisions taken by the leader, each follower has some room to respond, i.e., $P_i(x), i = 1,\ldots,K$, is non-empty. Moreover, to ensure that the LBMIF problem is well posed we assume that $P_i(x), i = 1,\ldots,K$, is a point-to-point map. References [2,5] show the difficulties which may arise when the follower rational reaction set is not single-valued for all permissible x, in linear bilevel problems with one follower.

(e) Inducible region or feasible region of the leader

$$\mathrm{IR}_{\mathrm{LBMIF}} = \{(x, y_1,\ldots,y_K) : (x, y_1,\ldots,y_K) \in S,\ y_i \in P_i(x),\ i = 1,2,\ldots,K\}.$$

We also assume that $\mathrm{IR}_{\mathrm{LBMIF}}$ is non-empty to guarantee the existence of a solution to the LBMIF problem.

With these definitions, the LBMIF problem formulated in (1a)–(1d) can be written as:

$$\begin{array}{ll} \min\limits_{x,y_1,\ldots,y_K} & c^t x + \sum\limits_{i=1}^{K} d_i^t y_i, \\ \text{subject to:} & (x, y_1, \ldots, y_K) \in \mathrm{IR}_{\mathrm{LBMIF}}. \end{array} \tag{2}$$

3 Transforming the LBMIF problem into a linear bilevel problem with one follower

Let us consider the following linear bilevel problem with one follower:
LB:

$$\min_{x,y_1,\ldots,y_K} \quad c^t x + \sum_{i=1}^{K} d_i^t y_i, \tag{3a}$$

$$\text{subject to:} \qquad Ax + \sum_{i=1}^{K} B_i y_i \le b_0, \tag{3b}$$

where $(y_1, \ldots, y_K)$ solves

$$\min_{y_1,\ldots,y_K} \quad \sum_{i=1}^{K} (v_i^t x + w_i^t y_i), \tag{3c}$$

$$\text{subject to:} \qquad Q_i x + D_i y_i \le b_i, \ i = 1, \ldots, K. \tag{3d}$$

Let S_{LB} denote its constraint set. Notice that $S_{\mathrm{LB}} = S$, the constraint set of the LBMIF problem. For a given x, let $S_{\mathrm{LB}}(x)$ denote the feasible set of the second level problem, i.e.,

$$S_{\mathrm{LB}}(x) = \{(y_1, \ldots, y_K) : D_i y_i \le b_i - Q_i x, \ i = 1, \ldots, K\}.$$

Similarly, let $P_{\mathrm{LB}}(x)$ the rational reaction set of the second level decision maker, i.e.,

$$P_{\mathrm{LB}}(x) = \operatorname*{argmin}_{y_1,\ldots,y_K} \left\{ \sum_{i=1}^{K} (v_i^t x + w_i^t y_i) : (y_1, \ldots, y_K) \in S_{\mathrm{LB}}(x) \right\}$$

and let $\mathrm{IR}_{\mathrm{LB}}$ denote the inducible region

$$\mathrm{IR}_{\mathrm{LB}} = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, \ (y_1, \ldots, y_K) \in P_{\mathrm{LB}}(x)\}.$$

Hence, the LB problem formulated in (3a)–(3d) can be written as:

$$\begin{array}{ll} \min\limits_{x,y_1,\ldots,y_K} & c^t x + \sum\limits_{i=1}^{K} d_i^t y_i, \\ \text{subject to:} & (x, y_1, \ldots, y_K) \in \mathrm{IR}_{\mathrm{LB}}. \end{array} \tag{4}$$

Theorem 3.1 *The LBMIF problem is equivalent to problem LB.*

Proof Since objective functions of problems (2) and (4) are equal, we only need to prove that $\mathrm{IR}_{\mathrm{LBMIF}} = \mathrm{IR}_{\mathrm{LB}}$.

Let $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \in \mathrm{IR}_{\mathrm{LBMIF}}$. Hence, $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \in S$ and $\bar{y}_i \in P_i(\bar{x}), i = 1, \ldots, K$.

Let us take the second level problem of LB for $x = \bar{x}$. If $(\bar{y}_1, \ldots, \bar{y}_K)$ is not an optimal solution for this problem, then $(y_1^*, \ldots, y_K^*) \in S_{\mathrm{LB}}(\bar{x})$ exists so that

$$\sum_{i=1}^{K}(v_i^t\bar{x} + w_i^t y_i^*) < \sum_{i=1}^{K}(v_i^t\bar{x} + w_i^t\bar{y}_i). \tag{5}$$

On the other hand, since $(y_1^*, \ldots, y_K^*) \in S_{\mathrm{LB}}(\bar{x})$, then $y_i^* \in S_i(\bar{x}),\ i = 1, \ldots, K$. Hence

$$v_i^t\bar{x} + w_i^t\bar{y}_i \leq v_i^t\bar{x} + w_i^t y_i^*, \quad i = 1, \ldots, K,$$

which contradicts (5). As a consequence, $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \in \mathrm{IR}_{\mathrm{LB}}$.

Similarly, let $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \in \mathrm{IR}_{\mathrm{LB}}$. Hence, $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \in S$, $(\bar{y}_1, \ldots, \bar{y}_K) \in S_{\mathrm{LB}}(\bar{x})$ and $\bar{y}_i \in S_i(\bar{x}), i = 1, \ldots, K$.

If $j \in \{1, 2, \ldots, K\}$ exists so that $\bar{y}_j \notin P_j(\bar{x})$, then

$$v_j^t\bar{x} + w_j^t y_j^* < v_j^t\bar{x} + w_j^t\bar{y}_j, \tag{6}$$

where y_j^* refers to the optimal solution of the jth follower second level problem for $x = \bar{x}$. On the other hand, since $(\bar{y}_1, \ldots, y_j^*, \ldots, \bar{y}_K) \in S_{\mathrm{LB}}(\bar{x})$ and $(\bar{y}_1, \ldots, \bar{y}_K)$ is an optimal solution to the second level problem of LB for $x = \bar{x}$, then

$$\sum_{i=1}^{K}(v_i^t\bar{x} + w_i^t\bar{y}_i) \leq \sum_{i=1}^{K} v_i^t\bar{x} + w_1^t\bar{y}_1 + \cdots + w_j^t y_j^* + \cdots + w_K^t\bar{y}_K$$

so, $v_j^t\bar{x} + w_j^t\bar{y}_j \leq v_j^t\bar{x} + w_j^t y_j^*$, which contradicts (6). As a consequence, $(\bar{x}, \bar{y}_1, \ldots, \bar{y}_K) \in \mathrm{IR}_{\mathrm{LBMIF}}$. □

This theorem allows us to apply all the results obtained and the algorithms developed for linear bilevel problems with one follower [2,5] to the multiple follower case, when the followers are independent. In particular, we know that the inducible region of the LBMIF problem is equal to the union of faces of S. As a consequence, a solution to the LBMIF problem occurs at a vertex of $\mathrm{IR}_{\mathrm{LBMIF}}$, thus a vertex of S, which can be obtained, for instance, by applying the kth-best algorithm.

4 Examining the definition of inducible region in [8]

In the final part of the paper we consider the results stated in a paper by Shi et al. [8] regarding the LBMIF problem. They conclude that an optimal solution to the LBMIF problem occurs at an extreme point of what they define as inducible region of the LBMIF problem. Next, we show with an example that this definition does not reflect the real inducible region since it is obtained by moving the first level constraints into each second level follower problem.

Shi et al. [8] define the feasible set for the ith follower for each x as:

$$S_i(x)_{\mathrm{SZL}} = \{y_i : (x, y_1, \ldots, y_K) \in S\}. \tag{7}$$

In words, the feasible set of each follower is constructed by projecting S onto the corresponding follower decision space. But S includes first level constraints which only affect the decision process of the leader, not the follower ones. Notice the paradoxical fact that the feasible region of each follower depends on the variables of the other

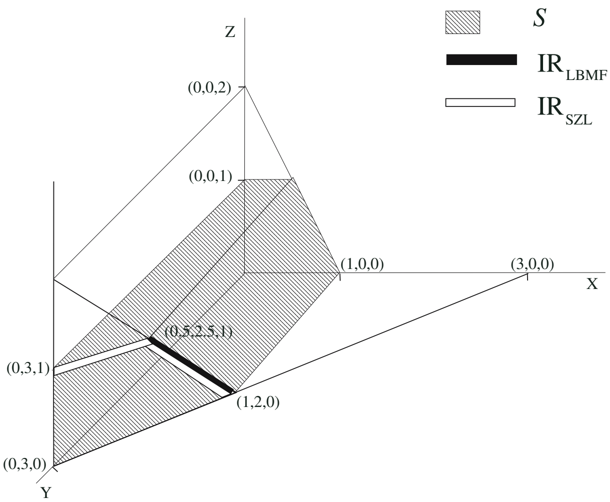

Fig. 1 Constraint region S and inducible region

followers, although the statement of the problem clearly indicates that the followers do not share any information. As a consequence, Shi et al. define the inducible region as:

$$\begin{aligned} \mathrm{IR_{SZL}} = \Big\{ & (x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, \\ & w_i^t y_i = \min[w_i^t \tilde{y}_i : B_i \tilde{y}_i \leq b_0 - Ax - \sum_{j=1, j \neq i}^{K} B_j y_j, \\ & \quad D_i \tilde{y}_i \leq b_i - Q_i x,\ D_j y_j \leq b_j - Q_j x, j = 1, \ldots, K, j \neq i],\ i = 1, \ldots, K \Big\} \end{aligned}$$

The following very simple example with one leader and two followers shows that to move first level constraints into the second level, as done by Shi, et al. and consider all constraints to define the feasible set of each follower provides a problem which is not equivalent to the original LBMIF problem (see Fig. 1).

$$\begin{aligned} \min_{x,y,z} \quad & x, \\ \text{s.t.} \quad & z \leq 1, \\ & x \geq 0, \end{aligned}$$

where y solves

$$\begin{aligned} \min_{y} \quad & -y, \\ \text{s.t.} \quad & x + y \leq 3, \\ & y \geq 0, \end{aligned}$$

where z solves

$$\begin{aligned} \min_{z} \quad & -z, \\ \text{s.t.} \quad & 2x + z \leq 2, \\ & z \geq 0. \end{aligned}$$

The constraint region is

$$S = \{(x, y, z) \in \mathbb{R}^3 : z \leq 1,\ x + y \leq 3,\ 2x + z \leq 2,\ x \geq 0,\ y \geq 0,\ z \geq 0\}.$$

For each $x \in S(X)$, the follower feasible sets are

$$S_1(x) = \{y \in \mathbb{R} : y \leq 3 - x,\ y \geq 0\}, \qquad S_2(x) = \{z \in \mathbb{R} : z \leq 2 - 2x,\ z \geq 0\}.$$

Hence, the inducible region is:

$$\mathrm{IR}_{\mathrm{LBMIF}} = \{(x, y, z) \in \mathbb{R}^3 : x \in [0.5, 1],\ y = 3 - x,\ z = 2 - 2x\}.$$

The optimal solution to the LBMIF problem is $(0.5, 2.5, 1)$ and the optimal objective function value is 0.5.

On the contrary,

$$\begin{aligned}
S_1(x)_{\mathrm{SZL}} &= \{y \in \mathbb{R} : x \geq 0,\ y \leq 3 - x,\ y \geq 0,\ z \leq 1,\ 2x + z \leq 2,\ z \geq 0\},\\
S_2(x)_{\mathrm{SZL}} &= \{z \in \mathbb{R} : x \geq 0,\ x + y \leq 3,\ y \geq 0,\ z \leq 1,\ z \leq 2 - 2x,\ z \geq 0\},\\
\mathrm{IR}_{\mathrm{SZL}} &= \begin{cases} (x, 3 - x, 1) & \text{if } x \in [0, 0.5],\\ (x, 3 - x, 2 - 2x) & \text{if } x \in [0.5, 1]. \end{cases}
\end{aligned}$$

Hence, the optimal solution would be $(0, 3, 1)$ and the optimal objective function value would be 0. Notice that $(0, 3, 1)$ is not even a feasible point of the LBMIF problem since for $x = 0$ the optimal solution to the second follower problem is $z = 2$.

For the linear bilevel problems with one follower, difficulties caused by moving first level constraints involving second level variables into the second level have already been stated in Refs. [1,5,7].

On the other hand, notice that, in the example, the 'misconstructed' inducible region of Ref. [8] contains the true inducible region as a subset. This fact is always true.

Remark 4.1 $\mathrm{IR}_{\mathrm{LBMIF}} \subset \mathrm{IR}_{\mathrm{SZL}}$.

Let $(\hat{x}, \hat{y}_1, \ldots, \hat{y}_K) \in \mathrm{IR}_{\mathrm{LBMIF}}$, then $\hat{y}_i$, $i = 1, \ldots, K$ is an optimal solution of the ith second level problem:

$$\begin{aligned}
&\min_{y_i} \quad v_i^t \hat{x} + w_i^t y_i,\\
&\text{subject to:} \qquad D_i y_i \leq b_i - Q_i \hat{x}.
\end{aligned}$$

Moreover, $(\hat{x}, \hat{y}_1, \ldots, \hat{y}_K)$ verifies first level constraints, i.e., $A\hat{x} + \sum_{i=1}^{K} B_i \hat{y}_i \leq b_0$. Hence, it immediately follows that $\hat{y}_i$, $i = 1, \ldots, K$ is an optimal solution of the problem

$$\begin{aligned}
&\min_{y_i} \quad w_i^t y_i,\\
&\text{subject to:} \quad B_i y_i \leq b_0 - A\hat{x} - \sum_{j=1, j\neq i}^{K} B_j \hat{y}_j,\\
&\qquad\qquad\qquad D_i y_i \leq b_i - Q_i \hat{x},\\
&\qquad\qquad\qquad D_j \hat{y}_j \leq b_j - Q_j \hat{x}, \quad j = 1, \ldots, K, \quad j \neq i.
\end{aligned}$$

Thus, $(\hat{x}, \hat{y}_1, \ldots, \hat{y}_K) \in \mathrm{IR}_{\mathrm{SZL}}$.

Similarly to Sect. 3, next we prove that the problem in Ref. [8] is equivalent to the following linear bilevel problem with one follower:

$\widetilde{\text{LB}}$:

$$\min_{x,y_1,\ldots,y_K} \quad c^t x + \sum_{i=1}^{K} d_i^t y_i, \qquad \text{where } (y_1,\ldots,y_K) \text{ solves} \tag{8a}$$

$$\min_{y_1,\ldots,y_K} \quad \sum_{i=1}^{K} (v_i^t x + w_i^t y_i), \tag{8b}$$

$$\text{subject to:} \qquad Ax + \sum_{i=1}^{K} B_i y_i \le b_0, \tag{8c}$$

$$Q_i x + D_i y_i \le b_i, \quad i = 1,\ldots,K. \tag{8d}$$

Notice that its constraint set is S. Moreover, for a given x, we denote by $S_{\widetilde{\text{LB}}}(x)$ the feasible set of the second level problem and by $\text{IR}_{\widetilde{\text{LB}}}$ the inducible region.

Theorem 4.2 *The problem*

$$\begin{array}{ll} \min\limits_{x,y_1,\ldots,y_K} & c^t x + \sum\limits_{i=1}^{K} d_i^t y_i, \\ \textit{subject to:} & (x, y_1,\ldots,y_K) \in IR_{\text{SZL}} \end{array}$$

is equivalent to the $\widetilde{LB}$ problem.

Proof Since both objective functions are the same, we only need to prove that $\text{IR}_{\text{SZL}} = \text{IR}_{\widetilde{\text{LB}}}$.

Let $(\bar{x},\bar{y}_1,\ldots,\bar{y}_K) \in \text{IR}_{\text{SZL}}$. If $(\bar{x},\bar{y}_1,\ldots,\bar{y}_K) \notin \text{IR}_{\widetilde{\text{LB}}}$ then $(y_1^*,\ldots,y_K^*) \in S_{\widetilde{\text{LB}}}(\bar{x})$ exists so that

$$\sum_{i=1}^{K} (v_i^t \bar{x} + w_i^t y_i^*) < \sum_{i=1}^{K} (v_i^t \bar{x} + w_i^t \bar{y}_i). \tag{9}$$

On the other hand, since $(y_1^*,\ldots,y_K^*) \in S_{\widetilde{\text{LB}}}(\bar{x})$, then $y_i^* \in S_i(\bar{x})_{\text{SZL}}$, $i = 1,\ldots,K$. Moreover $\bar{y}_i$ solves the ith follower problem, hence

$$v_i^t \bar{x} + w_i^t \bar{y}_i \le v_i^t \bar{x} + w_i^t y_i^*, \quad i = 1,\ldots,K,$$

which contradicts (9). As a consequence, $(\bar{x},\bar{y}_1,\ldots,\bar{y}_K) \in \text{IR}_{\widetilde{\text{LB}}}$.

Similar arguments demonstrate that $\text{IR}_{\widetilde{\text{LB}}} \subseteq \text{IR}_{\text{SZL}}$. Indeed, let $(\bar{x},\bar{y}_1,\ldots,\bar{y}_K) \in \text{IR}_{\widetilde{\text{LB}}}$. Hence, $(\bar{x},\bar{y}_1,\ldots,\bar{y}_K) \in S$, $(\bar{y}_1,\ldots,\bar{y}_K) \in S_{\widetilde{\text{LB}}}(\bar{x})$ and $\bar{y}_i \in S_i(\bar{x})_{\text{SZL}}$, $i = 1,\ldots,K$.

If $j \in \{1,2,\ldots,K\}$ exists so that $\bar{y}_j$ does not solve the jth follower problem, then

$$v_j^t \bar{x} + w_j^t y_j^* < v_j^t \bar{x} + w_j^t \bar{y}_j, \tag{10}$$

where y_j^* refers to the optimal solution of the jth follower problem for $x = \bar{x}$. On the other hand, since $(\bar{y}_1,\ldots,y_j^*,\ldots,\bar{y}_K) \in S(\bar{x})_{\widetilde{\text{LB}}}$ and $(\bar{y}_1,\ldots,\bar{y}_K)$ is an optimal solution to the second level problem of $\widetilde{\text{LB}}$ for $x = \bar{x}$, then

$$\sum_{i=1}^{K} (v_i^t \bar{x} + w_i^t \bar{y}_i) \le \sum_{i=1}^{K} v_i^t \bar{x} + w_1^t \bar{y}_1 + \cdots + w_j^t y_j^* + \cdots + w_K^t \bar{y}_K$$

so, $v_j^t \bar{x} + w_j^t \bar{y}_j \le v_j^t \bar{x} + w_j^t y_j^*$, which contradicts (10). As a consequence, $(\bar{x},\bar{y}_1,\ldots,\bar{y}_K) \in \text{IR}_{\text{SZL}}$. □

Remark 4.3

Theorem 4.2 provides us with another way of showing that Ref. [8] does not deal with the real LBMIF problem. Taking into account the problem equivalences proved throughout this paper, Shi et al. actually solve problem $\widetilde{\text{LB}}$ defined in (8a)–(8d), which is obtained by moving the first level constraints of problem LB defined in (3a)–(3d) into the second level. As previously mentioned, papers [1,5,7] show, for linear bilevel problems with one leader and one follower, that the problem obtained by transferring first level constraints which depend on variables of the second level, into the second level is not equivalent to the original one.

5 Conclusions

In this paper, we have analyzed the linear bilevel multi-follower programming problem with independent followers. By independent followers we mean that the objective function and the set of constraints of each follower only include the variables controlled by the leader and his own variables. We have proved that this problem can be transformed into a linear bilevel problem with only one follower. The second level objective function of the new problem is the sum of the follower objective functions. The feasible set for the second level consists of the whole set of constraints of all followers. Taking into account this result all theory and algorithms developed for linear bilevel problems can be directly applied to the problem analyzed in the paper. In particular, we can assert that an optimal solution of the problem occurs at an extreme point of the constraint region, which can be obtained by applying the kth-best algorithm. An obvious future work is to analyze this property when objective functions are not linear. On the other hand, we have shown that the way in which Shi et al. [8] tackle the LBMIF problem boils down to the transference of the first level constraints into each second level follower problem. This provides a misconstructed inducible region which contains the true inducible region as a subset.

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